MATH 117 Test 1 Solutions
Dr. Fenton Fall 2007

Individual portion

1. (7 pts.) The function $f(x) = \dfrac{x}{4-x^2}$ is an odd function. Prove this. Then sketch a graph of the function.
   Calculate $f(-x)$ as follows:

$$f(-x) = \frac{(-x)}{4-(-x)^2} = \frac{-x}{4-x^2} = -\frac{x}{4-x^2} = -f(x)$$

   Therefore f is an odd function. The graph has vertical asymptotes at $x = 2$ and $x = -2$, and is symmetric around the origin. (Sorry, I don't yet know how to include a graph in this type of file.)

2. (7 pts.) What is the domain of the function $g(x) = \sqrt{\ln(3x+2)}$? Give an exact answer.
   We must have $\ln(3x+2) \geq 0$, so $3x+2 \geq 1$. This gives $x \geq -\frac{1}{3}$.

3. (7 pts.) For $f(x) = \dfrac{4x-1}{2x+3}$, find an expression for $f^{-1}(x)$.
   Set this equal to y and solve for x, then switch variables. The result should be $f^{-1}(x) = \dfrac{1+3x}{4-2x}$.

4. (7 pts.) Find an equation that describes this parabola. (The picture showed an upside-down parabola with its vertex at $(-1, 3)$ and passing through the point $(0, 0)$.)
   This parabola is translated left 1, up 3, and is reflected downwards. It is also stretched vertically. So the equation has to fit the pattern $y = -a(x+1)^2 + 3$. To make sure $(0, 0)$ is on the curve, we need $0 = -a(0+1)^2 + 3$, so $a = 3$. Thus the equation is $y = -3(x+1)^2 + 3$.

5. (7 pts.) Let $f(x) = \cos^2(x)$ and $g(x) = 3x + 1$.
   (a) Find the composition $(f \circ g)(x)$.
   (b) Find new functions $F(x)$ and $G(x)$ so that $(F \circ G)(x) = (f \circ g)(x)$.

   (a) $(f \circ g)(x) = f(3x+1) = \cos^2(3x+1)$.
   (b) $F(x) = x^2$ and $G(x) = \cos(3x+1)$, which makes $(F \circ G)(x) = (f \circ g)(x)$.

6. (7 pts.) On the Moon, gravity is weaker than on the Earth. An object dropped from rest will fall $s(t) = 1.62t^2$ meters in t seconds.
   (a) Find the average velocity between $t = 1$ second and $t = 2$ seconds.
   (b) Find the average velocity between $t = 1$ second and $t = 1.1$ seconds.
   (c) Find the average velocity between $t = 1$ second and $t = 1 + h$ seconds.
   (d) Use your result from part (c) to find the instantaneous velocity at $t = 1$ second.

   (a) $s(1) = 1.62$ meters and $s(2) = 6.48$ meters. So $\dfrac{s(2) - s(1)}{2-1} = 4.86$ meters per second.
   (b) $s(1.1) = 1.9602$ meters. So $\dfrac{s(1.1) - s(1)}{1.1 - 1} = 3.402$ meters per second.
   (c) $s(1+h) = 1.62(1+h)^2$ meters. So $\dfrac{s(1+h) - s(1)}{(1+h) - 1} = 3.24 + 1.62h$ meters per second.
   (d) Instantaneous velocity $= \lim\limits_{h \to 0} 3.24 + 1.62h = 3.24$ meters per second.

7. (8 pts.) For the function h given by the graph below, state the value of each quantity, or tell why it does not exist.
   (a) $\lim_{x\to -3^-} h(x) = 2$
   (b) $\lim_{x\to -3^+} h(x) = 2$
   (c) $\lim_{x\to -3} h(x) = 2$ because the left- and right-hand limits agree.
   (d) $h(-3)$ is undefined; there is a hole in the graph. (The picture is not as clear as I desired, so 0 is an acceptable answer.)
   (e) $\lim_{x\to 0^-} h(x) = -2$
   (f) $\lim_{x\to 0^+} h(x) = 2$
   (g) $\lim_{x\to 0} h(x)$ does not exist because the the left- and right-hand limits do not agree.
   (h) $\lim_{x\to 5^-} h(x)$ does not exist because the function is oscillating too wildly.
   (i) $\lim_{x\to 5^+} h(x) = -\infty$. (The limit does not exist.)

Group portion

1. (7 pts.) Is this function even, odd, or neither? Show how you decide this.

$$f(x) = \frac{\sqrt{1+x^2} - |x|}{\cos(\frac{x}{5}) + 1}$$

   Calculate $f(-x)$ as follows:

$$f(-x) = \frac{\sqrt{1+(-x)^2} - |-x|}{\cos(\frac{(-x)}{5}) + 1} = \frac{\sqrt{1+x^2} - |x|}{\cos(\frac{x}{5}) + 1} = f(x)$$

   So f is an even function.

2. (7 pts.) If a bacteria population starts with 100 bacteria and doubles every three hours, then in an ideal environment the number of bacteria after t hours will be approximately $f(t) = 100e^{0.23t}$. How long will it take for the population to reach 50,000 bacteria?
   We need to solve $50000 = 100e^{0.23t}$. This reduces to $500 = e^{0.23t}$. Using the natural logarithm, $\ln 500 = 0.23t$, so $t = \frac{\ln 500}{0.23} \approx 27$ hours.

3. (9 pts.) Evaluate the following limit by hand.

$$\lim_{t\to 0} \frac{t}{\sqrt{1+t} - 1}$$

   The key is to rationalize the denominator.

$$\lim_{t\to 0} \frac{t}{\sqrt{1+t} - 1} \cdot \frac{\sqrt{1+t} + 1}{\sqrt{1+t} + 1} = \lim_{t\to 0} \sqrt{1+t} + 1 = 2$$

4. (9 pts.) Consider the function $f(x) = x - \dfrac{4}{x}$.

(a) Explain why this is not a one-to-one function.

(b) Restrict the domain of f so that it becomes a one-to-one function. What is the range of this new function?

(c) Draw the restricted function f and its inverse f^{-1} on the same set of coordinate axes.

(d) What are the domain and range of f^{-1}?

(a) One method is to look at a graph and use the Horizontal Line test. Antoher is to compute some values and see repeated outputs. For instance, $f(2) = 0 = f(-2)$.

(b) The easiest way to restrict the domain is to choose $x > 0$. The graph shows that the right side of the curve, by itself, is one-to-one. The range is $(-\infty, \infty)$.

(c) The graph of f^{-1} is the reflection of the graph of f across the diagonal line $y = x$. (Sorry, I don't know how to include the graph in this file.)

(d) The domain of f^{-1} is $-\infty < x < \infty$ and the range is $0 < y < \infty$.

5. (9 pts.) Consider $\lim_{x \to 2^+} \dfrac{x - c}{x - 2}$. What is the value of this limit if $c > 2$? What is the value if $c < 2$? What is the value if $c = 2$?
If $c > 2$, the limit gives $\frac{2-c}{0}$, which will be either $+\infty$ or $-\infty$. Since the numerator is negative and the denominator is positive as $x \to 2^+$, the limit is $-\infty$.
If $c < 2$, the limit again gives $\frac{2-c}{0}$. Since the numerator is now positive and the denominator is positive as $x \to 2^+$, the limit is $+\infty$.
If $c = 2$, the function becomes $\frac{x-2}{x-2}$, which gives a limit of 1.

6. (9 pts.) The statement $\lim_{x \to 2} \dfrac{6}{x} = 3$ means that $|\frac{6}{x} - 3| < \epsilon$ whenever $0 < |x - 2| < \delta$. If we pick $\epsilon = \frac{1}{3}$, what should be the corresponding value of δ?
Include a graph in your explanation.
The statement $|\frac{6}{x} - 3| < \frac{1}{3}$ can be re-written as $-\frac{1}{3} < \frac{6}{x} - 3 < \frac{1}{3}$. Solving this for x gives $1.8 < x < 2.25$. Since $2 - 1.8 = 0.2$ and $2.25 - 2 = 0.25$, and we must pick smaller than both of these, choose $\delta = 0.2$.