MATH 117 Test 3 Solutions
Dr. Fenton Fall 2007

1. (10 pts.) Find $\lim_{x\to 0} \dfrac{1-\cos(x)}{x^2}$.
ANSWER: This is in the pattern $\frac{0}{0}$ so use L'Hopital's Rule (twice, in fact).

$$\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{\sin(x)}{2x} = \lim_{x\to 0} \frac{\cos(x)}{2} = \frac{1}{2}$$

2. (10 pts.) Ohm's Law for electrical circuits says that $V = IR$, where V is the voltage, I is the current in amperes (amps), and R is the resistance in ohms. Suppose that a circuit measures 4.5 volts, 2.5 amps, and that the current is decreasing at 0.05 amps per minute while the resistance does not change. At what rate is the voltage changing?
ANSWER: With $V = 4.5$ and $I = 2.5$, we must have $R = \frac{4.5}{2.5} = 1.8$ ohms. The problem is asking for a rate of change, so we need to take the derivative.

$$V = IR \longrightarrow \frac{dV}{dt} = I \cdot \frac{dR}{dt} + R \cdot \frac{dI}{dt} = 2.5 \cdot 0 + 1.8 \cdot (-0.05) = -0.09$$

3. (10 pts.) Find the absolute maximum and the absolute minimum of the function $f(x) = 10 + 15x - 2x^2 - x^3$ on the interval $-1 \le x \le 3$.
ANSWER: To find absolute extrema, it is necessary to check the critical points *and* the end points of the given domain. Since $f'(x) = 15 - 4x - 3x^3 = (5-3x)(3+x)$, the critical values are $x = \frac{5}{3}$ and $x = -3$. Of course, -3 lies outside the interval.

   $f(-1) = -6$. This is the absolute minimum.

   $f(\frac{5}{3}) = \frac{670}{27} \approx 24.8$. This is the absolute maximum.

   $f(3) = 10$

4. (10 pts.) The Mean Value Theorem says that if a function f satisfies the hypotheses
   (a) f is continuous on the closed interval $[a, b]$ and
   (b) f is differentiable on the open interval (a, b)

   then there is a number c for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
Suppose that $f(x)$ is the function $x^3 + x - 1$. Verify that f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 2]$. Then find all values c that satisfy the conclusion of the theorem.
ANSWER: Since polynomial functions are always continuous, f is continuous on this interval. The derivative $f'(x) = 3x^2 + 1$ is always defined, f is differentiable on this interval. To find c we must solve

$$3c^2 + 1 = \frac{f(2) - f(0)}{2 - 0} = \frac{9 - (-1)}{2 - 0} = 5$$

   The only solution that lies in this interval is $c = \frac{2}{\sqrt{3}}$.

5. (10 pts.) A 100 mg sample of cesium-137 decayed to 89 mg in five years. What is the half-life of cesium-137?
ANSWER: We must solve $50 = 100e^{kt}$ for t. First, however, we need to know the value of k. So solve the equation $89 = 100e^{k\cdot 5}$. This gives $0.89 = e^{5k}$, which implies $5k = \ln(0.89)$. Thus $k = \frac{1}{5}\ln(0.890 \approx -0.0233$.
Now we can solve $50 = 100e^{-0.0233t}$ in a similar way to get the half-life equaling 29.75 years.

6. (10 pts.) Find the derivative of $y = (\cos(x))^x$.
ANSWER: This requires a logarithm to separate the exponent from the base.

$$\ln(y) = x \cdot \ln(\cos(x))$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{-\sin(x)}{\cos(x)} + 1 \cdot \ln(\cos(x))$$

$$\frac{dy}{dx} = (\cos(x))^x \left(-x \cdot \tan(x) + \ln(\cos(x))\right)$$

7. (10 pts.) Suppose that a particle moves according to the function $s(t) = t^3 - 10t^2 + 36t$, where t is measured in seconds and s is measured in cm. Find an expression for the velocity of this particle. What is the particle's minimum velocity on the interval $0 \leq t \leq 10$?
ANSWER: The velocity is $v(t) = 3t^2 - 20t + 36$. To find the minimum of velocity, we must check the critical values of v, which come fromn $v'(t) = 6t - 20$. The sole critical value is $t = \frac{10}{3}$. Since $v(0) = 36$, $v(10) = 136$, and $v(\frac{10}{3}) = \frac{8}{3}$, the minimum velocity is $\frac{8}{3}$ cm per second.

8. (10 pts.) Let $F(x) = \sin(2x) - x$ on the interval $[0, \pi]$. Find any critical values for F in this interval. Then use the second derivative test to decide whether each is a local maximum or local minimum.
ANSWER: $F'(x) = 2\cos(2x) - 1$. This equals 0 when $\cos(2x) = \frac{1}{2}$, which happens when $2x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$. Thus the critical values are $x = \frac{\pi}{6} \approx 0.534$ and $x = \frac{5\pi}{6} \approx 2.618$.
The second derivative is $F''(x) = -2\sin(2x)$. Since $F''(\frac{\pi}{6}) = -\sqrt{3}$, the graph is concave down at this value, making $x = \frac{\pi}{6}$ a local maximum. Also, since $F''(\frac{5\pi}{6}) = \sqrt{3}$, the graph is concave up at this value, making $x = \frac{\pi}{6}$ a local minimum.

9. (20 pts.) Consider the function $g(x) = \dfrac{14}{x^2+3} - 2$. Create a hand-drawn graph of this function that takes into account the following items.
   (a) domain
   (b) intercepts
   (c) symmetry
   (d) asymptotes
   (e) local extrema
   (f) increasing versus decreasing
   (g) concavity
   (h) inflection points

   It will be useful to know that $\dfrac{dy}{dx} = \dfrac{-28x}{(x^2+3)^2}$ and that $\dfrac{d^2y}{dx^2} = \dfrac{-84(1-x^2)}{(x^2+3)^3}$
   ANSWER: (I still do not know how to include pictures in these files; sorry!)
   (a) The domain is the entire set of real numbers.
   (b) intercepts: at $(0, \frac{8}{3})$ and at $(\pm 2, 0)$.
   (c) symmetry: even because $g(-x) = g(x)$.
   (d) asymptotes: since the demonimator never equals zero, there are no vertical asymptotes. Because $\lim_{x\to\infty} g(x) = -2$, there is a horizontal asymptote at $y = -2$.
   (e) local extrema: The first derivative has only one critical value, at $x = 0$. The point $(0, \frac{8}{3})$ is a local maximum (which we can see from increasing/decreasing).
   (f) increasing versus decreasing: When $x < 0$ the first derivative is positive so $g(x)$ is increasing. When $x > 0$ the first derivative is negative so $g(x)$ is decreasing.

(g) concavity: The second derivative has critical values at ± 1. When $x < -1$ and when $x > 1$, the second derivative is positive so the graph is concave up. For $-1 \leq x \leq 1$, the second derivative is negative so the graph is concave down,

(h) inflection points: at $(\pm 1, \frac{3}{2})$.

The intercept points, the extreme points, and the inflection points are important for the graph.