MATH 120 Dr. Fenton Fall 2010

Test 1, Individual portion solutions

1. (10 pts.) Here are some numbers. Put them in increasing order (smallest to largest). Show how you do this.

$$46.\overline{8},\ 57.4_{eight},\ 143.2_{five},\ 101110.1_{two}$$

One way to do this is to convert every number to base ten.

$$57.4_{eight} = 5 \cdot 8 + 7 + \frac{4}{8} = 47.5$$

$$143.2_{five} = 1 \cdot 25 + 4 \cdot 5 + 3 + \frac{2}{5} = 48.4$$

$$101110.1_{two} = 32 + 8 + 4 + 2 + \frac{1}{2} = 46.5$$

So the correct order is $101010.01_{two},\ 46.\overline{8}, 57.4_{eight},\ 143.2_{five}$.

2. (15 pts.) Is $q \Rightarrow (p \vee r)$ equivalent to $\neg p \vee q \vee \neg r$? Use truth tables to decide.

p	q	r	$q \Rightarrow (p \vee r)$	$\neg p \vee q \vee \neg r$
true	true	true	true $\Rightarrow$ (true $\vee$ true) = true	$\neg$ true $\vee$ true $\vee$ $\neg$ true = true
true	true	false	true $\Rightarrow$ (true $\vee$ false) = true	$\neg$ true $\vee$ true $\vee$ $\neg$ false = true
true	false	true	false $\Rightarrow$ (true $\vee$ true) = true	$\neg$ true $\vee$ false $\vee$ $\neg$ true = false
true	false	false	false $\Rightarrow$ (true $\vee$ false) = true	$\neg$ true $\vee$ false $\vee$ $\neg$ false = true
false	true	true	true $\Rightarrow$ (false $\vee$ true) = true	$\neg$ false $\vee$ true $\vee$ $\neg$ true = true
false	true	false	true $\Rightarrow$ (false $\vee$ false) = false	$\neg$ false $\vee$ true $\vee$ $\neg$ false = true
false	false	true	false $\Rightarrow$ (false $\vee$ true) = true	$\neg$ false $\vee$ false $\vee$ $\neg$ true = true
false	false	false	false $\Rightarrow$ (false $\vee$ false) = true	$\neg$ false $\vee$ false $\vee$ $\neg$ false = true

Since the last two columns disagree in the third and sixth rows, the two expressions are not equal.

3. (15 pts.) Find $gcd(736, 598)$ and $lcm(736, 598)$ using one of the methods discussed in class. Show how you do this.
Following the Euclidean Algorithm,

$$736 = 1 \cdot 598 + 138$$

$$598 = 4 \cdot 138 + 46$$

$$138 = 3 \cdot 46 + 0$$

So $gcd(736, 598) = 46$. For $lcm(736, 598)$, calculate $\frac{736 \cdot 598}{46} = 9568$.

4. (10 pts.)
   (a) Write the decimal $1.\overline{297}$ as a fraction in lowest terms.
   (b) Write $\frac{30}{11}$ as a repeating decimal.

   (a) Let $x = 1.\overline{297}$. The repeating block is three digits long, so multiply x by 1000 to get $1000x = 1297.\overline{297}$. Subtracting the two equations gives $999x = 1296$. Therefore $x = \frac{1296}{999} = \frac{48}{37}$.
   (b) Doing the long division gives $\frac{30}{11} = 2.\overline{72}$.

5. (15 pts.) "If n mod $6 = 0$, then n mod $2 = 0$ and n mod $3 = 0$."
   (a) Write the negation of this statement, without using an implication.

(b) Write the contrapositive of the original statement.

(c) Decide which is true, the negation or the contrapositive, and explain why.

(a) The negation is "$n \bmod 6 = 0$, and $n \bmod 2 \neq 0$ or $n \bmod 3 \neq 0$." This requires DeMorgan's Laws.

(b) The contrapositive is "If $n \bmod 2 \neq 0$ or $n \bmod 3 \neq 0$, then $n \bmod 6 \neq 0$."

(c) The contrapositive is true because a value n that is not divisible by 2 or that is not divisible by 3 cannot be divisible by 6. (Besides, the contrapositive means the same thing as the original statement, and a number n that is divisible by 6 must also be divisible by 2 and by 3.)

6. (10 pts.) For the equation below, find all solutions in the mod 10 number system.

$$x^2 + 6x \equiv 5 (\text{mod } 10)$$

The best way to find these solutions is to try all the values in the *mod* 10 number system, which are $0, 1, \ldots, 9$. For instance, using $x = 9$ gives $9^2 + 6 \cdot 9 = 135$, which in *mod* 10 equals 5. The only other solution is $x = 5$.

7. (15 pts.) Here is an arithmetic sequence, defined as a recursive function.

$$A(n) = \begin{cases} -10 & \text{if } n = 1 \\ A(n-1) + 3 & \text{if } n > 1 \end{cases}$$

(a) Write the first ten terms of this sequence.

(b) Calculate $A(50)$. Show how you do this.

(a) The pattern of the sequence is to start with -10 then add 3 to get each new term. So the first ten terms are $-10, -7, -4, -1, 2, 5, 8, 11, 14, 17$.

(b) Instead of listing fifty terms of the sequence, we can start with -10 and think of adding 3 forty nine times. So $A(50) = -10 + 49 \cdot 3 = 137$.

8. (10 pts.) Is it true that $|\lfloor x \rfloor| \geq \lfloor |x| \rfloor$? Explain why or why not.
Yes, this is true. When $x \geq 0$, the absolute value has no effect and the two quantities are equal. When the number x is negative, the floor function will round down to a smaller value. The absooute value will then create a larger value. For example, $|\lfloor -1.5 \rfloor| = |-2| = 2$ but $\lfloor |-1.5| \rfloor = \lfloor 1.5 \rfloor = 1$.

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Test 1, Group portion solutions

1. (8 pts.) Find $gcd(10430, 14938)$ using the Euclidean Algorithm. Then find $lcm(10430, 14938)$. Show how you do this. Following the Euclidean Algorithm,

$$14938 = 1 \cdot 10430 + 4508$$
$$10430 = 2 \cdot 4508 + 1414$$
$$4508 = 3 \cdot 1414 + 266$$
$$1414 = 5 \cdot 266 + 84$$
$$266 = 3 \cdot 84 + 14$$
$$84 = 6 \cdot 14 + 0$$

So $gcd(10430, 14938) = 14$. For the least common multiple, calculate $\frac{14938 \cdot 10430}{14} = 11128810$.

2. (8 pts.) The *biconditional* statement $p \Leftrightarrow q$, is defined to mean $(p \Rightarrow q) \wedge (q \Rightarrow p)$.
   (a) Use truth tables to show that this is equivalent to $(p \wedge q) \vee (\neg p \wedge \neg q)$.
   (b) Find the negation of the statement in part (a). Simplify your answer.
   (c) Explain in English what your answer to part (b) means.

   (a) Here are the truth tables.

p	q	$(p \Rightarrow q) \wedge (q \Rightarrow p)$	$(p \wedge q) \vee (\neg p \wedge \neg q)$
true	true	(true ⇒ true) ∧ (true ⇒ true) = true	(true ∧ true) ∨ (false ∧ false) = true
true	false	(true ⇒ false) ∧ (false ⇒ true) = false	(true ∧ false) ∨ (false ∧ true) = false
false	true	(false ⇒ true) ∧ (true ⇒ false) = false	(false ∧ true) ∨ (true ∧ false) = false
false	false	(false ⇒ false) ∧ (false ⇒ false) = true	(false ∧ false) ∨ (true ∧ true) = true

   (b) The negation is $\neg((p \wedge q) \vee (\neg p \wedge \neg q))$. Using DeMorgan's Laws, this equals $\neg(p \wedge q) \wedge \neg(\neg p \wedge \neg q) = (\neg p \vee \neg q) \wedge (p \vee q)$.
   (c) The only way to make this negation true is for one of p and q to be false and one to be true. So p and q must have different values.

3. (8 pts.) "If $x + y$ is an odd number, then $x \cdot y$ is an even number."
   (a) Is this statement true or false? Explain why.
   (b) State the converse of this statement. Is the converse true or false? Explain why.

   (a) This statement is true. If the sum is odd, that means that one of x and y is even and the other is odd. The product $x \cdot y$ of even times odd will be even.
   (b) The converse of this statement is
   "If $x \cdot y$ is an even number, then $x + y$ is an odd number."
   This can be false. For example, $2 \cdot 4 = 8$ but $2 + 4 = 6$.

4. (8 pts.) In the mod 12 number system, solve the equation $x^2 + 2x \equiv 3 \pmod{12}$. Show how you do this.
   The best way to find these solutions is to try all the values in the *mod* 12 number system, which are

$0, 1, \ldots, 11$. For instance, using $x = 7$ gives $7^2 + 2 \cdot 7 = 63$, which equals 3 in mod 12. The other solutions are $x = 1, 3, 9$.

5. (8 pts.) A little boy was trying to arrange his toy soldiers in rows. When he tried making rows of four, he had one extra soldier. So he threw one away. Then he tried making rows of five from the remaining soldiers, and again he had one extra soldier. So he threw one more soldier away. Then he tried rows of six, and once again he had an extra soldier which he threw away. Finally, the soldiers he had left came out exactly right in rows of seven.
How many soldiers did he start with? Write a loop to help you decide.

You can use a loop to look for numbers that work. The *mod* command can be used to check for multiples of 4, of 5, etc. Be sure to reduce the number being checked for each step.

```
for x from 1 to 1000 do
if(x mod 4 = 1 and (x-1) mod 5 = 1 and (x-2) mod 6 = 1 and (x-3) mod 7 = 0, x, NULL)
end do
```

The smallest value produced by this loop is 297. (The next value is 717, which is a LOT of toy soldiers.)

6. (10 pts.) Here is a recursive function.

$$f(n) = \begin{cases} 1 & \text{if } n = 1 \\ f(n-1) + 2^{n-1} & \text{if } n > 1 \end{cases}$$

(a) Create this function in Maple.

(b) Create a sequence showing at least the first ten output values of the function f.

(c) Write a closed-form expression for a function that produces the same values as f.

(a) One way to do the Maple code is
$f := n \rightarrow piecewise(n = 1, 1, n > 1, f(n-1) + 2^{n-1})$

(b) With the command $seq(f(n), n = 1..10)$ you will get $1, 3, 7, 15, 31, 63, 127, 255, 511, 1023$.

(c) A non-recursive function to produce the same values as f could be $F(n) = 2^n - 1$.