

Problems 1 - 10 were on both the individual portion and the group portion. Problems 11 and 12 were on only the group portion.

1. Suppose that A is a subset of the natural numbers. Are the following sentences negations of each other? Explain why or why not.

“All numbers in A are even.”

“All even numbers are in A .”

No, these sentences are not negations of each other. The first sentence is $(\forall x \in A)(x \text{ is even})$. Its negation is $(\exists x \in A)(x \text{ is odd})$, which has the wrong quantifier for the second sentence.

A couple people pointed out that the set $A = \{2, 4, 6, 8, 10, \dots\}$ makes both sentences true. So they cannot be negations of each other.

2. Here is the beginning of a sequence: $[2, 5, 8, 11, 14, \dots]$

- (a) What kind of sequence is this?
(b) Write a recursive function for this sequence.

This is an arithmetic sequence because each term is obtained by adding 3 to the previous term. One way to write this as a recursive function is

$$f(n) = \begin{cases} 2 & \text{if } n = 1 \\ f(n-1) + 3 & \text{if } n \geq 2 \end{cases}$$

3. Let S and T be two sets. Does $(S^c \cap T)^c$ equal $(S \cap T^c)$? Draw Venn diagrams to prove your answer. No, $(S^c \cap T)^c$ does not equal $(S \cap T^c)$. If we use DeMorgan's Laws on the first expression, $(S^c \cap T)^c = (S \cup T^c)$ because the operation in the middle changes. The Venn diagrams show the difference.

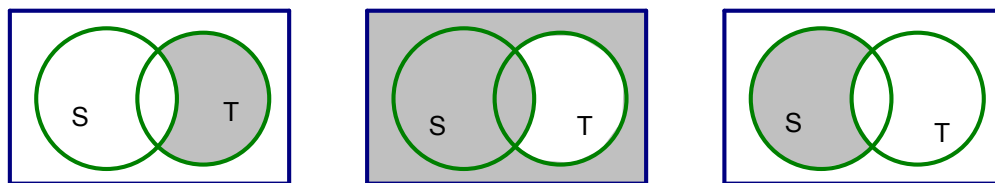


Figure 1: $S^c \cap T$

$(S^c \cap T)^c$

$S \cap T^c$

4. A function is said to be *onto* if $(\forall y \in \text{Target})(\exists x \in \text{Domain})(y = f(x))$. Consider the function $f(x) = \lfloor \frac{x}{3} \rfloor$ with the target $\{-5, \dots, 5\}$. Find a domain that makes this function onto. Show why your answer is correct.

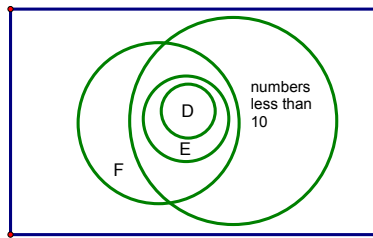
There are many correct choices for the domain D . Here is a simple one: $D = \{-15, -12, -9, -6, -3, 0, 3, 6, 9, 12, 15\}$.

Since $f(-15) = \lfloor \frac{-15}{3} \rfloor = \lfloor -5 \rfloor = -5$, $f(-12) = \lfloor \frac{-12}{3} \rfloor = \lfloor -4 \rfloor = -4$, etc., all of the values in the target can be gotten as outputs from these inputs.

5. Suppose that D, E, F are nonempty sets of integers, with $D \subset E \subset F$. Also suppose that $(\forall x \in E)(x > 10)$. Which of the following statements is definitely true? Which might be true? Which is definitely false? Justify your answers. Include a Venn diagram in your explanation.

- $(\exists x \in D)(x > 10)$
- $(\forall x \in F)(x > 10)$
- $(\forall x \in D)(x \leq 10)$
- $(\exists x \in F)(x \leq 10)$

- $(\exists x \in D)(x > 10)$ This is definitely true. All values of D are in E so are greater than 10.
- $(\forall x \in F)(x > 10)$ This might be true. The elements of E are greater than 10 but there are elements of $F \setminus E$ that we don't know about.
- $(\forall x \in D)(x \leq 10)$ This is definitely false. Values from D also belong in E , and so these values must be greater than 10. (Besides, this statement is the negation of the first statement in this list.)
- $(\exists x \in F)(x \leq 10)$ This might be true. Again, there are elements of $F \setminus E$ that we don't know about.



6. Suppose that S is a subset of the integers. Consider the following statement:

$$(\exists x \in S)(\forall y \in S)(x + y \text{ is even})$$

- (a) Give an example of a set S that makes this statement true. Explain why your example works.
- (b) Now give an example of a set S that makes the negation of this statement true. Explain why this example works.

- (a) The set $S = \{2, 4\}$ makes this statement true. Let $x = 2$. Then $2 + 2$ and $2 + 4$ are both even.
- (b) The negation of the statement is $(\forall x \in S)(\exists y \in S)(x + y \text{ is odd})$. The set $S = \{1, 2\}$ makes the negation true. When $x = 1$, use $y = 2$. When $x = 2$, use $y = 1$. (Actually, the empty set would work as another example!)

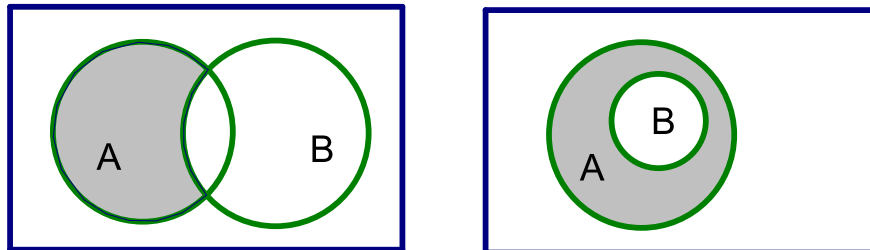
7. Here are two sets. Find the cardinality of $P \cup Q$.

P = the set of two-digit natural numbers that start with an odd digit

Q = the set of two-digit natural numbers that end with an odd digit

Use the inclusion-exclusion formula: $|P \cup Q| = |P| + |Q| - |P \cap Q|$. The set P contains the 10s, the 30s, the 50s, the 70s, and the 90s, so $|P| = 50$. The set Q contains 5 values from the 10s, 5 values from the 20s, etc. so $|Q| = 9 \cdot 5 = 45$. For $|P \cap Q|$, look at the elements they have in common. These are 11, 13, 15, 17, 19, 31, 33, ..., 97, 99, and there are 25 of these values. So $|P \cup Q| = 50 + 45 - 25 = 70$.

8. For nonempty sets A and B , when does $|A \setminus B| = |A| - |B|$? Explain why your answer is correct. The only time this works is when B is a subset of A . Otherwise $|A \setminus B|$ is greater than $|A| - |B|$.



9. Which of the following sets has the greatest cardinality? Show how you decide this.

$$\mathbf{A} := \{p \mid p < 40 \wedge \text{isprime}(p)\}$$

$$\mathbf{B} := \{[x, y] \mid x \in \{1, 2, 3\} \wedge y \in \{2, 4, 6, 8, 10\}\}$$

$$\mathbf{C} := \{S \mid S \subseteq \{10, 20, 30, 40\}\}$$

- $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37\}$ so $|A| = 12$.
- Since there are three options for the first coordinate and five options for the second coordinate, $|B| = 3 \cdot 5 = 15$.
- Since $\{10, 20, 30, 40\}$ has four elements, there are $2^4 = 16$ subsets and $|C| = 16$.

Therefore C has the greatest cardinality.

10. Use the four steps of mathematical induction to prove that

$$(\forall n \in \text{Natural numbers}) \left(1^2 + 2^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6} \right)$$

Predicate: $1^2 + 2^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$

Base case: When $n = 1$, $1^2 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6}$.

Implication: If $1^2 + 2^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$ then $1^2 + 2^2 + \dots + (n+1)^2 = \frac{(n+1) \cdot (n+2) \cdot (2n+3)}{6}$.

Proof:

$$1^2 + 2^2 + \dots + n^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

$$1^2 + 2^2 + \dots + n^2 + (n+1)^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6} + (n+1)^2$$

$$1^2 + 2^2 + \dots + n^2 + (n+1)^2 = (n+1) \cdot \frac{2n^2 + 7n + 6}{6}$$

$$1^2 + 2^2 + \dots + n^2 + (n+1)^2 = \frac{(n+1) \cdot (n+2) \cdot (2n+3)}{6}$$

11. Does $(\exists x \in \mathcal{N})(P(x) \wedge Q(x))$ mean the same thing as $(\exists x \in \mathcal{N})P(x) \wedge (\exists x \in \mathcal{N})Q(x)$? (The set \mathcal{N} is the set of natural numbers.) Explain why or why not. Include examples in your explanations.
- No, these do not mean the same thing. The first says that there is a single x -value that makes both $P(x)$ and $Q(x)$ true. The second says that there is a value to make $P(x)$ true and that there is a value (maybe a different value) to make $Q(x)$ true. For example, $(\exists x \in \mathcal{N})(x \text{ is even} \wedge x \text{ is odd})$ is false, since no number can be both even and odd. However, $((\exists x \in \mathcal{N})(x \text{ is even}) \wedge (\exists x \in \mathcal{N})(x \text{ is odd}))$ is true because 2 is even and 3 is odd.

12. Here is a recursive sequence:

$$G(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ G(n-1) \cdot G(n-2) & \text{if } n \geq 3 \end{cases}$$

Use mathematical induction to prove that, for every natural number n ,

$$G(n) = 2^{F(n)}$$

where $F(n)$ is the n th Fibonacci number.

Because the recursion is two steps deep, this actually needs complete induction.

Predicate: $G(n) = 2^{F(n)}$

Base case: When $n = 1$, $G(1) = 1$ and $2^{F(1)} = 2^0 = 1$.

Implication: If $(\forall k \leq n)(G(k) = 2^{F(k)})$ then $G(n+1) = 2^{F(n+1)}$.

Proof:

$$G(n+1) = G(n) \cdot G(n-1) = 2^{F(n)} \cdot 2^{F(n-1)} = 2^{F(n)+F(n-1)} = 2^{F(n+1)}$$