

1. Let  $A$  and  $B$  be two sets. Does  $(A \cap B^c)^c$  equal  $(A^c \cup B)$ ? Draw Venn diagrams to prove your answer. Yes,  $(A \cap B^c)^c = (A^c \cup B)$ . This relies on one of DeMorgan's Laws.

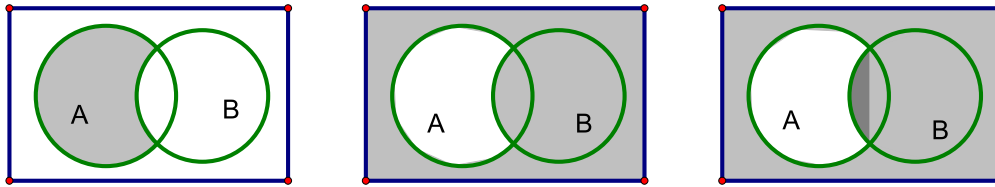


Figure 1:  $A \cap B^c$

$(A \cap B^c)^c$

$A^c \cup B$

2. Are the following sentences negations of each other? Explain why or why not.

“I have a friend who does not have a cell phone.”

“All of my friends have cell phones.”

Yes, these are negations of each other. The first sentence is

$$(\exists x \in \text{my friends})(x \text{ does not have a cell phone})$$

. Its negation is  $(\forall x \in \text{my friends})(x \text{ has a cell phone})$ .

3. Suppose that  $S$  is a subset of the real numbers. Consider the following statement:

$$(\exists x \in S)(\forall y \in S)(y \leq x)$$

- (a) Write the negation of this statement and simplify your answer.  
 (b) Give an example of a set  $S$  that makes the original statement true. Explain your example.
- (a) The negation is  $(\forall x \in S)(\exists y \in S)(y > x)$   
 (b) There are many correct examples. Any set with a largest element will work. Here is one:  $S = \{0, 1\}$ .

4. Here is a recursive function defined on the domain of natural numbers.

$$f(n) = \begin{cases} 2 & \text{if } n = 1 \\ 3 + f(n - 1) & \text{if } n > 1 \end{cases}$$

- (a) Find the first five outputs of the function  $f$ .  
 (b) Write an closed-form expression for  $f$ . Explain why your answer is correct.
- (a)  $f(1) = 2$ ,  $f(2) = 3 + 2 = 5$ ,  $f(3) = 3 + 5 = 8$ ,  $f(4) = 3 + 8 = 11$ , and  $f(5) = 3 + 11 = 14$ .

(b)  $f(n) = 2 + 3(n - 1) = 3n - 1$ . The values increase by 3 each time, so it is a linear function with slope 3. The value -1 gives the correct starting value.

5. (a) Give an example of sets  $R$ ,  $S$ , and  $T$  so that  $R \subset \{4\} \subset S \subset T \subset \{1, 2, 3, 4, 5\}$ .

(b) Now give an example of sets  $X$  and  $Y$  for which  $X \subseteq \{1, 2, 3, 4, 5\}$  and  $Y \subseteq \{1, 2, 3, 4, 5\}$ , but  $X$  is not a subset of  $Y$  and  $Y$  is not a subset of  $X$ .

(a) There are several possibilities. However,  $R$  must be the empty set. Here is one answer.

$$\{\} \subset \{4\} \subset \{1, 4\} \subset \{1, 2, 4\} \subset \{1, 2, 3, 4, 5\}$$

(b) Again, there are several correct answers. One answer is  $X = \{1, 2, 3\}$  and  $Y = \{2, 3, 4\}$ .

6. Here is a function that uses subsets from  $\{1, 2, \dots, 10\}$  as inputs.

$$F(S) = \max(S)$$

In other words, the function  $F$  returns the maximum value in the set  $S$ .

(a) The target for  $F$  is the set  $\{1, 2, \dots, 10\}$ . Is  $F$  onto? Explain why or why not.

(b) Is  $F$  one-to-one? Explain why or why not.

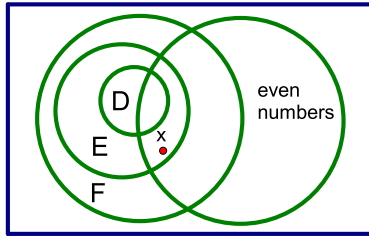
(a) Yes,  $F$  is onto.  $F(\{1\}) = 1$ ,  $F(\{2\}) = 2$ , etc.

(b) No,  $F$  is not one-to-one.  $F(\{1, 2, 10\}) = F(\{3, 4, 10\})$ .

7. Suppose that  $D, E, F$  are nonempty sets of integers, with  $D \subset E \subset F$ . Also suppose that  $(\exists x \in E)(x \text{ is even})$ . Which of the following statements is definitely true? Which might be true? Which is definitely false? Justify your answers. Include a Venn diagram in your explanation.

- $(\exists x \in D)(x \text{ is even})$
- $(\exists x \in F)(x \text{ is even})$
- $(\forall x \in D)(x \text{ is odd})$
- $(\forall x \in F)(x \text{ is odd})$

- $(\exists x \in D)(x \text{ is even})$  This might be true, but we cannot be sure where the even value from  $E$  lies.
- $(\exists x \in F)(x \text{ is even})$  This is definitely true. Any even  $x$  from  $E$  also lies in  $F$ .
- $(\forall x \in D)(x \text{ is odd})$  This might be true. It depends on where the even values from  $E$  lie.
- $(\forall x \in F)(x \text{ is odd})$  This is definitely false. The even values in  $E$  make it false.



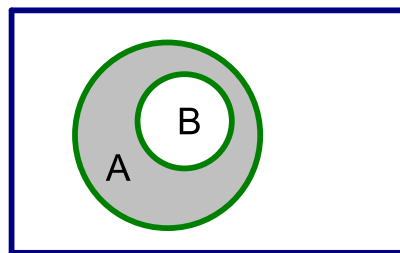
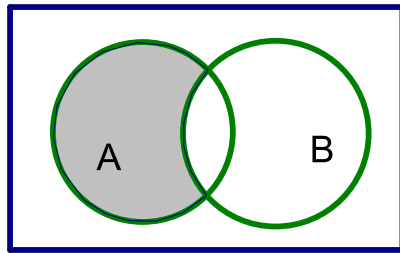
8. Here are two sets. Find the cardinality of  $P \cup Q$ .

$$P = \left\{ \frac{1}{50}, \frac{2}{50}, \frac{3}{50}, \frac{4}{50}, \dots, \frac{49}{50} \right\}$$

$$Q = \left\{ \frac{1}{75}, \frac{2}{75}, \frac{3}{75}, \frac{4}{75}, \dots, \frac{74}{75} \right\}$$

Use the inclusion-exclusion formula:  $|P \cup Q| = |P| + |Q| - |P \cap Q|$ . Clearly  $|P| = 49$  and  $|Q| = 74$ . For  $|P \cap Q|$ , look at the elements they have in common. These are  $\frac{2}{50} = \frac{3}{75} = \frac{1}{25}$ ,  $\frac{4}{50} = \frac{6}{75} = \frac{2}{25}$ , etc., and there are 24 of them. So  $|P \cup Q| = 49 + 74 - 24 = 99$ .

9. For nonempty sets  $A$  and  $B$ , when does  $|A \setminus B| = |A| - |B|$ ? Explain why your answer is correct. The only time this works is when  $B$  is a subset of  $A$ . Otherwise  $|A \setminus B|$  is greater than  $|A| - |B|$ .



10. Use the four steps of mathematical induction to prove that

$$(\forall n \in \text{Natural numbers}) (1 + 3 + \dots + (2n - 1) = n^2)$$

Predicate:  $1 + 3 + \dots + (2n - 1) = n^2$

Base case: When  $n = 1$ ,  $1 = 1^2$ .

Implication: If  $1 + 3 + \dots + (2n - 1) = n^2$  then  $1 + 3 + \dots + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$ .

Proof:

$$1 + 3 + \dots + (2n - 1) = n^2$$

$$1 + 3 + \dots + (2n - 1) + (2n + 1) = n^2 + (2n + 1)$$

$$1 + 3 + \dots + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$$

**Extra Credit** (5 pts.) Is the following statement true or false, and why?

$$(\forall x \in \text{Integers}) (\forall y \in \text{Integers}) ((x - y) \bmod 7 = 0 \Rightarrow (y - x) \bmod 7 = 0)$$

This is true. If any two integers  $x$  and  $y$  make the quantity  $(x - y)$  divisible by 7, then its negative  $(y - x)$  is also divisible by 7.